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Self-Scheduled $\mathcal{H}_\infty$ Control of Missile via Linear Matrix Inequalities

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This paper is concerned with the application of advanced linear parameter-varying (LPV) techniques to the global control of a missile. The LPV technique considered in this paper is an extension of the standard $\mathcal{H}_\infty$ synthesis technique to the case where the plant depends affinely on a time-varying vector $\theta(t)$. Working in the class of LPV plants, the proposed methodology produces an LPV controller, that is, a controller that is automatically "gain scheduled" along the trajectories of the plant. LPV controller solutions to the problem are characterized via a set of Riccati linear matrix inequalities that can be solved using convex programming. The missile under consideration is highly nonlinear and nonstationary. Its LPV model exhibits brutal parameter variations as functions of the flight conditions (angle of attack, speed, altitude). Additionally, some measurements are corrupted by flexible modes and the performance requirements are very strong. Since they obliterate the LPV nature of the plant, usual linear time-invariant techniques may be helpless for this problem. The power and advantages of the proposed methodology as an efficient tool to handle the global performance and robustness of the missile over its entire operating range are demonstrated.

I. Introduction and Motivations

THREE important classes of linear systems can be investigated: 1) *Linear time-invariant* (*LTI*) systems comprise the more familiar class in the control system literature and give rise to the most successful and self-contained theories. LTI plants are described in state-space form as

$$\dot{x} = Ax + Bu, \qquad y = Cx + Du$$

2) *Linear time-varying* (*LTV*) systems are systems for which the theory is not as mature as for LTI systems and consequently potential applications are more restricted to special problems. The state-space description of LTV systems is completely defined by the functional time dependence of the state-space data $A(t)$, $B(t)$, $C(t)$, and $D(t)$, whose associated differential equations are

$$\dot{x} = A(t)x + B(t)u, \qquad y = C(t)x + D(t)u$$

3) *Linear parameter-varying* (*LPV*) systems are linear systems where the state-space entries $A(\cdot)$, $B(\cdot)$, $C(\cdot)$, and $D(\cdot)$ are now explicit functions of a time-varying parameter $\theta(t)$. They have therefore a differential representation in the form

$$\begin{aligned} \dot{x} &= A[\theta(t)]x + B[\theta(t)]u \\ y &= C[\theta(t)]x + D[\theta(t)]u \end{aligned} \tag{1}$$

It also must be pointed out that an LPV system reduces to an LTV system for a given trajectory $\theta := \theta(t)$ and is transformed into an LTI system on constant trajectories $\theta := \theta_0 \ \forall t \geq 0$. It follows that for a frozen value of the parameter, LPV systems can be analyzed as LTI systems. Though the LPV properties (i.e., with θ time varying) of an LPV system cannot generally be inferred from its underlying LTI properties,[1,2] they provide useful tools that are part of the engineering practice.

Despite formal analogies, LTI, LTV, and LPV systems have fundamental differences. One of the most important is the "off-line" or "in-line" nature of such systems. Specifically, the LTI and LTV classes are composed of off-line systems in the sense that the state-space data A, B, C, and D or $A(t)$, $B(t)$, $C(t)$, and $D(t)$ must be known beforehand. In contrast, LPV systems are in-line systems as they are completely known when the trajectory $\theta := \theta(t)$ is known, that is, when the plant is operating and experiences a particular trajectory in its domain. In other words, the study of LPV systems requires the anticipation of the future behavior of the plant and thus has immediate advantages for real-world applications. Another notion intimately related to LPV systems is that of an operating domain. The operating domain Θ of an LPV system (1) is the parameter range of the system. This will be expressed by the inclusion relation $\theta(t) \in \Theta$ in the sequel. Summing up the above remarks, an LPV system is well defined whenever its parameter dependence and its operating domain are fixed. Moreover, LPV systems can be given at least three interesting interpretations:

1) They can be viewed as linear systems subject to uncertainties $\theta(t)$. As a consequence an adequate controller must be robust in the face of parameter uncertainties. The controller $K(s)$ is generally LTI and one has to solve a robust control problem.

2) They can be viewed as a family of linear systems derived from the linearization of a nonlinear plant or more simply as parameter-dependent systems where the parameter values are directly exploitable in the control structure. This leads to a scheduling problem in a general sense (see Fig. 1).

3) Mixing the above two situations, one comes up with the more general case where the parameter vector θ can be partitioned as $\theta := (\theta_1; \theta_2)^T$ where θ_1 is known in real time and θ_2 is an uncertain parameter. It must be emphasized that most practical control problems enter this last situation.

Therefore, the notion of LPV systems provide a formalization of various important issues of the control design problem. In our opinion, the derivation of LPV or self-scheduled controllers (see Fig. 1) is an essential part of the LPV framework. Such controllers have the same parameter dependence as the plant and can be described in state-space form as

$$\begin{aligned} \dot{x} &= A_K[\theta(t)]x + B_K[\theta(t)]y \\ u &= C_K[\theta(t)]x + D_K[\theta(t)]y \end{aligned}$$

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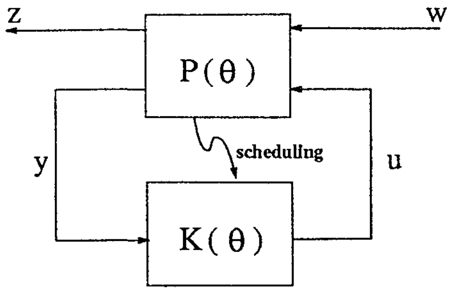

Fig. 1 $\mathcal{H}_\infty$ **synthesis structure for LPV systems.**

The improvements over classical LTI designs that can be expected from using this new control structure are stability and performance, robustness, and globality.

Stability and performance. Some LPV systems are not even stabilizable via a single LTI controller. The LPV control structure is potentially more powerful for solving such problems because it incorporates the parameter measurements. The same remarks apply to the performance objectives. Higher performance can generally be achieved with a controller that adjusts to the true plant dynamics.

Robustness. It is widely accepted that the design problem is primarily a trade-off between performance and robustness objectives. Since the nominal performance objectives are easier to satisfy with an LPV controller, it turns out that the compromise between performance and robustness is made more feasible with this class of controllers.

Globality. The classical approach to the control of LPV systems proceeds by 1) selecting a number of operating conditions of the LPV plant, 2) designing LTI controllers for the selected points, and 3) scheduling or interpolating the LTI controllers to derive the global control law of the original LPV plant. It must be pointed out that steps 1 and 2 are critical. As a result, the final control law, though working locally, does not provide any guarantee concerning the LPV behavior of the system, that is, when the parameter θ is really varying in time. The advantages of a synthesis technique working in the larger class of LPV systems appear clearly. It allows one to bypass the critical phases of the classical approach. Indeed, the whole operating range is handled in "one shot" without requiring repeated designs and scheduling.

From a practical point of view, the missile control design remains a very challenging task. Indeed, a missile is a nonlinear and rapidly parameter-varying plant operating in a large range of aerodynamic conditions. In addition to strong performance requirements, the design methodology has to meet two different kinds of objectives.

1) It must provide adequate margins (gain, phase, delay) and flexible mode attenuation when the system is analyzed as an LTI system.

2) It must handle a large operating domain and therefore must provide some kind of adjustment to the current plant dynamics.

Modern synthesis methodologies similar to those considered in Refs. 3–5 give a systematic answer to the first type of objectives. However, the gain-scheduling task becomes obscure because of the complexity (number of states) of the resulting controllers. In contrast, adaptive techniques[6,7] integrate the gain-scheduling aspects but are often inadequate to address LTI robustness specifications. In short, these techniques capture one particular design concern but fail to integrate both robust control and gain scheduling. The central idea leading to the development of LPV control techniques[8–14] is to address simultaneously both problems within the same design methodology.

II. Notations and Definitions

Throughout the paper, matrix transfer functions will be denoted by $P(s)$, where s is the Laplace variable. Here, $\mathcal{L}_2(\mathbb{R}^n)$ denotes the Hilbert space of the functions w mapping $\mathbb{R}_+$ into $\mathbb{R}^n$ with bounded $\mathcal{L}_2$ norm, that is,

$$\|w\|_2 := \sqrt{\int_0^\infty w(t)^T w(t)\,\mathrm{d}t} < \infty$$

The $\mathcal{L}_2$-induced norm of an operator T mapping $\mathcal{L}_2$ into $\mathcal{L}_2$ is defined as

$$\sup_{w\in\mathcal{L}_2} \frac{\|Tw\|_2}{\|w\|_2}$$

This norm coincides with the usual $\mathcal{H}_\infty$ norm when T is a linear time-invariant operator.

For real symmetric matrices M, the notation $M > 0$ means positive definite and indicates that all the eigenvalues of M are positive. Similarly, $M < 0$ means negative definite, that is, all the eigenvalues of M are negative.

Matrix polytopes are defined as the convex hull of a finite number of matrices N_i with the same dimensions, that is,

$$Co\{N_i\colon i = 1, \ldots, r\} := \left\{ \sum_{i=1}^{r} \alpha_i N_i\colon \alpha_i \geq 0, \sum_{i=1}^{r} \alpha_i = 1 \right\}$$

Finally, for a 2×2 block-partitioned matrix

$$P := \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$$

and a matrix K, the linear fractional transformation, if well posed, is defined as

$$F_l(P, K) := P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21}$$

III. $\mathcal{H}_\infty$ Control of LPV Systems: Brief Review

This section is a brief review of the main results of the $\mathcal{H}_\infty$ synthesis technique for LPV systems. The reader is referred to Refs. 14–16 for further details and complete solutions of both the continuous- and discrete-time problems.

A. Preliminary Notions

We are interested in a particular class of LPV systems, where 1) the parameter dependence is affine, that is, the state-space matrices $A[\theta(t)]$, $B[\theta(t)]$, $C[\theta(t)]$, $D[\theta(t)]$ depend affinely on $\theta(t)$, and, 2) the time-varying parameter $\theta(t)$ varies in a polytope Θ of vertices $\theta_1, \theta_2, \ldots, \theta_r$, that is,

$$\theta(t) \in \Theta := Co\{\theta_1, \theta_2, \ldots, \theta_r\}$$

These vertices represent the extremal values of the parameters. Though not fully general, this description encompasses many practical situations. From this characterization, it is clear that the state-space matrices $A[\theta(t)]$, $B[\theta(t)]$, $C[\theta(t)]$, $D[\theta(t)]$ evolve in a polytope of matrices whose vertices are the images of the vertices $\theta_1, \theta_2, \ldots, \theta_r$. In other words,

$$\begin{bmatrix} A(\theta) & B(\theta) \\ C(\theta) & D(\theta) \end{bmatrix} \in Co\left\{ \begin{bmatrix} A_i & B_i \\ C_i & D_i \end{bmatrix} : i = 1, \ldots, r \right\} \tag{2}$$

where

$$\begin{bmatrix} A_i & B_i \\ C_i & D_i \end{bmatrix} := \begin{bmatrix} A(\theta_i) & B(\theta_i) \\ C(\theta_i) & D(\theta_i) \end{bmatrix}$$

Because of this property, and with a slight abuse of language, we will refer to such LPV plants as "polytopic" in the sequel.

With these notations in mind, the $\mathcal{H}_\infty$ control problem for LPV systems parallels the customary $\mathcal{H}_\infty$ synthesis where the $\mathcal{H}_\infty$ norm bound is replaced by the more suitable notion of quadratic $\mathcal{H}_\infty$ performance and the plant under consideration is now LPV.

Definition 1 (quadratic $\mathcal{H}_\infty$ performance). The LPV system (1) has quadratic $\mathcal{H}_\infty$ performance γ if and only if there exists a Lyapunov function $V(x) = x^T X x$ with $X > 0$ that establishes global stability and the $\mathcal{L}_2$ gain of the input/output map is bounded by γ. That is,

$$\|y\|_2 < \gamma \|u\|_2$$

along all possible parameter trajectories $\theta(t)$ in Θ. □

B. Problem Description

We consider an augmented LPV plant mapping exogenous inputs

w and control inputs u to controlled outputs z and measured outputs y, i.e.,

$$
\begin{aligned}
\dot{x} &= A[\theta(t)]x + B_1[\theta(t)]w + B_2[\theta(t)]u \\
z &= C_1[\theta(t)]x + D_{11}[\theta(t)]w + D_{12}[\theta(t)]u \\
y &= C_2[\theta(t)]x + D_{21}[\theta(t)]w + D_{22}[\theta(t)]u \\
\theta(t) &\in \Theta := Co\{\theta_1, \theta_2, \dots, \theta_r\} \qquad \forall t \geq 0
\end{aligned} \tag{3}
$$

The plant is further assumed to be polytopic, i.e.,

$$
\begin{bmatrix} A[\theta(t)] & B_1[\theta(t)] & B_2[\theta(t)] \\ C_1[\theta(t)] & D_{11}[\theta(t)] & D_{12}[\theta(t)] \\ C_2[\theta(t)] & D_{21}[\theta(t)] & D_{22}[\theta(t)] \end{bmatrix} \in \mathcal{P}
$$

$$
:= Co\left\{\begin{bmatrix} A_i & B_{1i} & B_{2i} \\ C_{1i} & D_{11i} & D_{12i} \\ C_{2i} & D_{21i} & D_{22i} \end{bmatrix}, \; i = 1, 2, \dots, r\right\} \tag{4}
$$

where $A_i, B_{1i}, \dots$ denote the values of $A[\theta(t)], B_1[\theta(t)], \dots$ at the vertices $\theta(t) = \theta_i$ of the parameter polytope. The problem dimensions are given by

$$
A[\theta(t)] \in \mathbf{R}^{n \times n}, \qquad D_{11}[\theta(t)] \in \mathbf{R}^{p_1 \times m_1}
$$
$$
D_{22}[\theta(t)] \in \mathbf{R}^{p_2 \times m_2} \tag{5}
$$

The assumptions on the plant are as follows:

A1. $D_{22}[\theta(t)] = 0$ or equivalently $D_{22i} = 0$ for $i = 1, 2, \dots, r$.

A2. $B_2[\theta(t)]$, $C_2[\theta(t)]$, $D_{12}[\theta(t)]$, $D_{21}[\theta(t)]$ are parameter independent or, equivalently,

$$
B_{2i} = B_2, \qquad C_{2i} = C_2, \qquad D_{12i} = D_{12}, \qquad D_{21i} = D_{21} \qquad i = 1, 2, \dots, r \tag{6}
$$

A3. The pairs $(A(\theta), B_2)$ and $(A(\theta), C_2)$ are quadratically stabilizable and quadratically detectable over Θ, respectively.

It must be noted that assumptions A1 and A2 can be alleviated using very simple manipulations. See Ref. 14 for more details.

With these notations and assumptions in mind, the $\mathcal{H}_\infty$ control problem for LPV systems can be stated as follows.

Problem statement 1 (*$\mathcal{H}_\infty$ control problem of LPV systems*). Find an LPV Controller of the form

$$
\begin{aligned}
\dot{x} &= A_K[\theta(t)]x + B_K[\theta(t)]y \\
u &= C_K[\theta(t)]x + D_K[\theta(t)]y
\end{aligned}
$$

which guarantees quadratic $\mathcal{H}_\infty$ performance γ for the closed-loop system of Fig. 1 (see Definition 1). This will ensure that 1) the closed-loop system is quadratically stable over Θ and 2) the $\mathcal{L}_2$-induced norm of the operator mapping w into z is bounded by γ for all possible trajectories $\theta(t)$ in Θ. □

C. Characterization of Solutions

The following theorem taken from Ref. 14 presents complete solvability conditions for the $\mathcal{H}_\infty$ control problem of LPV systems in the form of a set of linear matrix inequalities (LMIs).

Theorem 1 (*existence conditions*). Consider a continuous LPV polytopic plant (3) and assume A1–A3. Let $\mathcal{N}_R$ and $\mathcal{N}_S$ denote bases of the null space of (B_2^T, D_{12}^T) and (C_2, D_{21}), respectively.

There exists an LPV controller guaranteeing quadratic $\mathcal{H}_\infty$ performance γ along all parameter trajectories in the polytope Θ if and only if there exist two symmetric matrices (R, S) in $\mathbf{R}^{n \times n}$ satisfying the system of $2r + 1$ linear matrix inequalities:

$$
\left[\begin{array}{c|c} \mathcal{N}_R & 0 \\ \hline 0 & I \end{array}\right]^T \left[\begin{array}{cc|c} A_i R + R A_i^T & R C_{1i}^T & B_{1i} \\ C_{1i} R & -\gamma I & D_{11i} \\ \hline B_{1i}^T & D_{11i}^T & -\gamma I \end{array}\right] \times \left[\begin{array}{c|c} \mathcal{N}_R & 0 \\ \hline 0 & I \end{array}\right] < 0 \qquad i = 1, \dots, r \tag{7}
$$

$$
\left[\begin{array}{c|c} \mathcal{N}_S & 0 \\ \hline 0 & I \end{array}\right]^T \left[\begin{array}{cc|c} A_i^T S + S A_i & S B_{1i} & C_{1i}^T \\ B_{1i}^T S & -\gamma I & D_{11i}^T \\ \hline C_{1i} & D_{11i} & -\gamma I \end{array}\right] \times \left[\begin{array}{c|c} \mathcal{N}_S & 0 \\ \hline 0 & I \end{array}\right] < 0 \qquad i = 1, \dots, r \tag{8}
$$

$$
\begin{bmatrix} R & I \\ I & S \end{bmatrix} \geq 0 \tag{9}
$$

□

This theorem only gives existence conditions. Since these conditions are LMIs in the variables R and S, they are convex and fall into the scope of efficient convex optimization techniques. For γ, R, and S solutions to the LMIs (7–9), there always exist LPV polytopic controllers solving the problem. In turn, such controllers are described by a system of LMIs from which one can extract a particular solution by simple algebraic manipulations.[14] More precisely, along some trajectory $\theta(t)$ in the polytope Θ, i.e.,

$$
\theta(t) = \sum_{i=1}^{r} \alpha_i(t)\theta_i \tag{10}
$$

the state-space matrices $A_K[\theta(t)]$, $B_K[\theta(t)]$, $C_K[\theta(t)]$, $D_K[\theta(t)]$ of the LPV polytopic controllers read

$$
\begin{bmatrix} A_K[\theta(t)] & B_K[\theta(t)] \\ C_K[\theta(t)] & D_K[\theta(t)] \end{bmatrix} := \sum_{i=1}^{r} \alpha_i(t) \begin{bmatrix} A_{Ki} & B_{Ki} \\ C_{Ki} & D_{Ki} \end{bmatrix} \tag{11}
$$

where the α_i are computed according to the convex decomposition (10).

It is important to note that whereas the state-space data A_{Ki}, B_{Ki}, C_{Ki}, D_{Ki} can be computed off-line, the LPV controller matrices $A_K(\theta)$, $B_K(\theta)$, $C_K(\theta)$, $D_K(\theta)$ must be updated in real time depending on the parameter measurement $\theta(t)$.

IV. Self-Scheduled $\mathcal{H}_\infty$ Control of Missile

This section presents a complete application of the LPV synthesis technique developed in Sec. III to the control of a missile pitch axis.

The missile dynamics are highly dependent upon the angle of attack α, the velocity V, and the altitude H. These three variables completely define the flight condition or operating point of the missile. They are assumed to be measured in real time. Therefore, based on the linearization of the missile equations around its flight conditions, an LPV representation can be developed. The self-scheduled $\mathcal{H}_\infty$ control presented above is then immediately applicable to this problem.

Before going further, an open-loop analysis of the missile pitch axis dynamics is presented. As a result of this preliminary analysis, the motivations for using LPV control appear clearly. Then following the problem description, the control law development and performance assessment will be presented.

A. Open-Loop Analysis

It is important to note that it is implicitly assumed that the pitch, yaw, and roll axes are decoupled. Though this assumption ignores some coupling phenomena of the missile, it greatly simplifies the design task and retains all the central (LPV) difficulties of the problem. The pitch axis model of the missile is depicted in the block diagram of Fig. 2.

The associated linearized dynamics of the missile (LPV part) are described by a state-space representation in the form

$$
\begin{aligned}
\begin{bmatrix} \dot{\alpha} \\ \dot{q} \end{bmatrix} &= \begin{bmatrix} -Z_\alpha & 1 \\ -M_\alpha & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ q \end{bmatrix} + \begin{bmatrix} 0 \\ M_{\delta_m} \end{bmatrix} \delta_m \\
\begin{bmatrix} a_z \\ q \end{bmatrix} &= \begin{bmatrix} -Z_\alpha V & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ q \end{bmatrix}
\end{aligned} \tag{12}
$$

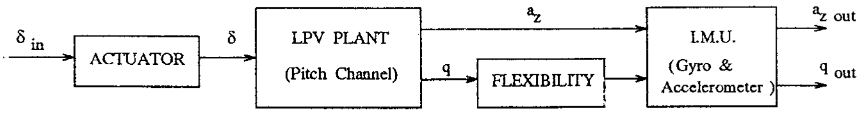

Fig. 2 Block diagram general description.

where α denotes the angle of attack, q is the pitch rate, a_z is the vertical acceleration, δ_m denotes the fin deflection, and V is the airspeed.

Note first that the missile dynamics do not include the lift force Z_δ due to the fin because, in this problem, it can be neglected as compared to the lift force due to the angle of attack. Another simplification concerns the flexibility that only affects the gyroscope. Generally speaking, the flexible modes appear both on gyroscope and accelerometer measurements. In our problem, however, the inertial measurement unit (IMU) has been designed in such a way that the flexibility weakly affects the accelerometer measurements and consequently can be removed from the problem.

The varying parameters Z_α, M_α, and M_{δ_m} are functions of the flight condition (α, V, H) and are therefore available for measurement in real time. It turns out that the problem can be further simplified by incorporating the parameter dependence of the input and measurement matrices into the LPV controller. This yields a simplified state-space LPV representation of the missile where the input and measurement matrices are no longer parameter dependent:

$$\begin{bmatrix} \dot{\alpha} \\ \dot{q} \end{bmatrix} = \begin{bmatrix} -Z_\alpha & 1 \\ -M_\alpha & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ q \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \delta_m$$
$$\begin{bmatrix} a_{zv} \\ q \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ q \end{bmatrix} \tag{13}$$

where a_{zv} is the normalized acceleration, i.e.,

$$a_{zv} = \frac{a_z}{Z_\alpha V}$$

In this simpler form the LPV description of the missile satisfies assumptions A1 and A2 and the proposed synthesis technique is directly applicable. Hereafter, the parameter vector of the LPV plant (13) is denoted as

$$\theta(t) := \begin{bmatrix} Z_\alpha(t) \\ M_\alpha(t) \end{bmatrix}$$

Although the parameter dependence of the original plant has been simplified, the control of the missile dynamics remains difficult. Indeed, the parameters M_α and Z_α abruptly change as functions of the flight condition and range over a large parameter domain where the stability properties of the missile are greatly influenced. Moreover, the system can switch between stability and high-instability regions depending on the sign of the parameter M_α. Analyzed as an LTI plant, the characteristic polynomial of the plant (13) reads $s^2 + Z_\alpha s + M_\alpha$. It follows that the plant is LTI unstable whenever M_α is negative. The parameter Z_α is less critical and influences the damping.

The missile speed varies between Mach 0.5 and Mach 4. The altitude belongs to the interval [0, 18,000] (meters) and the angle of attack evolves between 0 deg and ± 40 deg. Moreover, a small increase in the angle of attack may induce large parameter variations. The parameter range is then large and defined by $[-370, 380]$ for M_α and $[0.30, 4.40]$ for Z_α.

For future validation of the final LPV controller, 25 operating points have been selected in the parameter range. Figure 3 displays the poles of the corresponding 25 LTI models.

It appears clearly from this figure that the search for a single LTI controller providing adequate robustness and performance for all these operating conditions appears rather unfeasible. Actually, $\mathcal{H}_\infty$ or μ synthesis techniques fail to give an answer to that kind of problem. It is worth mentioning that these difficulties are enforced by the LPV or nonstationary nature of the missile. Even though an adequate LTI controller exists, it will not necessarily guarantee satisfying robustness and performances when the parameters are rapidly varying.[1,2,17]

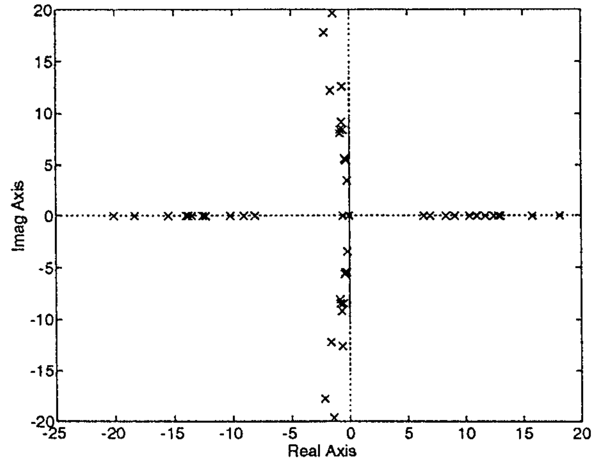

Fig. 3 Poles of LTI modes issued from LPV plant.

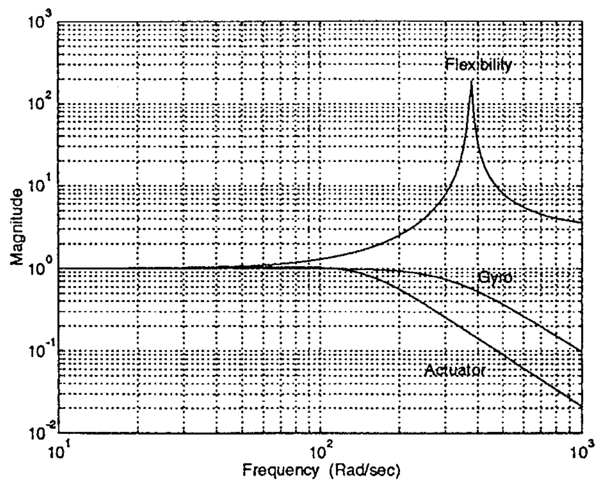

Fig. 4 Actuator, gyroscope, and flexible-mode frequency responses.

As already mentioned (see Fig. 2), in addition to the LPV plant dynamics, tail deflection actuators, gyroscopes, and bending flexible modes must be integrated into the overall model of the missile. The gyroscopes and actuators are adequately represented by second- and third-order transfer functions, respectively. Flexible modes are modeled as a multiplicative output LTI perturbation affecting the measurement of the pitch rate q. In fact, flexibility should be modeled as an LPV system, since its frequency is varying when the missile is operating. However, for simplification purposes, the worst case LTI situation has been considered, that is, the greatest peak at the lowest frequency. All resulting frequency responses are depicted in Fig. 4.

B. LPV Synthesis Structure

In this application, a two-degree-of-freedom synthesis structure is considered as in Fig. 5. This structure includes a feedforward part, $K_2(\theta)$, and a feedback part, $K_1(\theta)$, and is potentially more powerful to achieve strong performance requirements than the usual unity feedback structure.

The design procedure completely parallels usual $\mathcal{H}_\infty$ synthesis except that the operators to be minimized are now parameter dependent. Thus, the minimization must handle all possible trajectories in

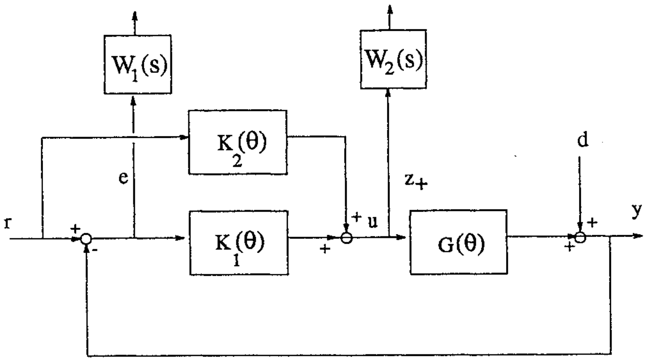

Fig. 5 Two-degree-of-freedom control structure.

the operating domain. Here, we consider a mixed-sensitivity problem suitably adapted to our particular control structure in Fig. 5. The performance objectives are expressed through the sensitivity operator $S(\theta)$ whereas additive robustness is captured by the operator $K_1(\theta)S(\theta)$. Note that setting $K_2(\theta) := 0$ in the interconnection of Fig. 5 and denoting $T_{a\to b}$ the operator mapping the signal a into the signal b yield the well-known relations

$$T_{r\to e}(\theta) = -T_{d\to e}(\theta) = S(\theta) := [I + G(\theta)K_1(\theta)]^{-1}$$

$$T_{r\to z_+}(\theta) = -T_{d\to z_+}(\theta) := K_1(\theta)S(\theta)$$

where all signals r, d, e, and z_+ are defined in Fig. 5. The self-scheduled $\mathcal{H}_\infty$ control problem consists of finding an LPV controller, denoted in operator form as

$$K(\theta) := [K_1(\theta)K_2(\theta)]$$

which, for all all admissible trajectories $\theta(t)$ in Θ, satisfies the following objectives: 1) internal stability of the closed-loop system of Fig. 5 and 2) minimization of the $\mathcal{L}_2$-induced gain of the closed-loop operator

$$\begin{bmatrix} W_1 & 0 \\ 0 & W_2 \end{bmatrix} F_l(P(\theta), K(\theta))$$

Here, W_1 and W_2 are weighting LTI operators and are described in the sequel.

The synthesis structure containing all necessary ingredients is depicted in Fig. 5. It is readily shown that the unweighted interconnection LPV plant $P(\theta)$ of Fig. 1 is completely defined by the state-space relations:

$$\begin{bmatrix} \dot{x} \\ z_+ \\ e \\ r \end{bmatrix} = \begin{bmatrix} A[\theta(t)] & 0 & 0 & B \\ 0 & 0 & 0 & I \\ -C & I & -I & 0 \\ 0 & I & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ r \\ d \\ u \end{bmatrix} \tag{14}$$

where the measured vector signal is

$$\begin{bmatrix} e \\ r \end{bmatrix} \qquad e := r - y$$

It turns out that the polytope of $P(\theta)$ is obtained as

$$Co\left\{\begin{bmatrix} A_i & 0 & 0 & B \\ 0 & 0 & 0 & I \\ -C & I & -I & 0 \\ 0 & I & 0 & 0 \end{bmatrix}, \; i = 1, \ldots, 4\right\}$$

where the A_i are the vertex matrices of $A(\theta)$ and are obtained as the extremal values of θ, i.e.,

$$\theta_1 := \begin{bmatrix} Z_\alpha^{\min} \\ M_\alpha^{\min} \end{bmatrix}, \qquad \theta_2 := \begin{bmatrix} Z_\alpha^{\max} \\ M_\alpha^{\min} \end{bmatrix}$$

$$\theta_3 := \begin{bmatrix} Z_\alpha^{\min} \\ M_\alpha^{\max} \end{bmatrix}, \qquad \theta_4 := \begin{bmatrix} Z_\alpha^{\max} \\ M_\alpha^{\max} \end{bmatrix}$$

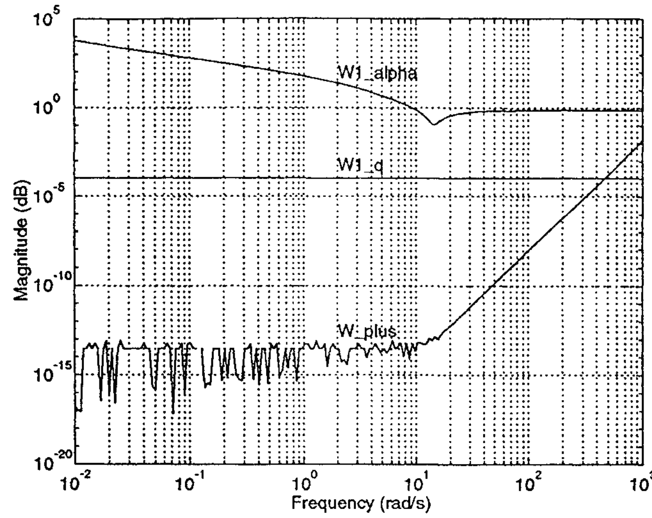

Fig. 6 Weighting functions.

The synthesis structure of Fig. 5 is readily obtained by weighting the performance signals e and z_+. The resulting interconnection structure is in turn LPV, and its associated polytope is described as the convex hull of the weighted vertices of $P(\theta)$.

Since assumptions A1–A2 are satisfied, the proposed LPV synthesis technique may now be applied for any given set of weighting functions $\{W_1(s), W_2(s)\}$ such that assumption A3 remains valid.

The selection of weight is based on a frozen-time analysis of the LPV system and follows the same lines as classical $\mathcal{H}_\infty$ synthesis. The robustness and performance requirements for the missile analyzed as an LTI plant around all possible operating conditions are the following:

1) Settling time is about 0.2 s and overshoot limitation is 15%.

2) High-frequency gain limitation is required for measurement noise and flexible mode attenuation.

3) Classical margin constraints are imposed on the closed-loop system. A gain margin of around 5 dB and a phase margin no worse than 30 deg are desirable.

Several trials were necessary to select adequate weighting filters. Finally, based on different closed-loop analyses, the following filters were adopted:

1) The sensitivity weight is described as

$$W_1(s) = \begin{bmatrix} W_{1\alpha}(s) & 0 \\ 0 & W_{1q}(s) \end{bmatrix}$$

where $W_{1\alpha}(s)$ is a second-order low-pass filter and $W_{1q}(s)$ is a simple static gain.

2) The robustness weight $W_2(s)$ is a sixth-order Chebyshev high-pass filter.

The corresponding frequency gain responses are shown in Fig. 6.

The LPV synthesis structure of Fig. 5 being fixed, the LMIs (7–9) are solved using LMI-Lab.[18] The software LMI-Lab is based on the projective techniques of Nesterov and Nemirovsky.[19] Many other interesting control applications of the LMI framework are developed Refs. 20 and 21 for nonlinear systems.

A performance level of $\gamma = 1.1$ was achieved after 100 iterations of the algorithm. Then, an LPV polytopic controller can be constructed [see Eq. (11)]. The formal expression of the LPV controller can be recovered by solving the convex decomposition problem (10) for the α_i in the case of a box. The following formulas for the state-space data of the LPV controller are easily obtained:

$$\begin{bmatrix} A_K\begin{bmatrix} Z_\alpha(t) \\ M_\alpha(t) \end{bmatrix} & B_K\begin{bmatrix} Z_\alpha(t) \\ M_\alpha(t) \end{bmatrix} \\ C_K\begin{bmatrix} Z_\alpha(t) \\ M_\alpha(t) \end{bmatrix} & A_K\begin{bmatrix} Z_\alpha(t) \\ M_\alpha(t) \end{bmatrix} \end{bmatrix} := \sum_{i=1}^{4} \alpha_i(t) \begin{bmatrix} A_{Ki} & B_{Ki} \\ C_{Ki} & D_{Ki} \end{bmatrix} \tag{15}$$

where

$$
\begin{aligned}
\alpha_1 &:= 1 - \frac{Z_\alpha - Z_\alpha^{\min}}{Z_\alpha^{\max} - Z_\alpha^{\min}} - \frac{M_\alpha - M_\alpha^{\min}}{M_\alpha^{\max} - M_\alpha^{\min}} \\
&\quad + \min\left(\frac{Z_\alpha - Z_\alpha^{\min}}{Z_\alpha^{\max} - Z_\alpha^{\min}}, \frac{M_\alpha - M_\alpha^{\min}}{M_\alpha^{\max} - M_\alpha^{\min}}\right) \\
\alpha_2 &:= \frac{Z_\alpha - Z_\alpha^{\min}}{Z_\alpha^{\max} - Z_\alpha^{\min}} - \min\left(\frac{Z_\alpha - Z_\alpha^{\min}}{Z_\alpha^{\max} - Z_\alpha^{\min}}, \frac{M_\alpha - M_\alpha^{\min}}{M_\alpha^{\max} - M_\alpha^{\min}}\right) \\
\alpha_3 &:= \frac{M_\alpha - M_\alpha^{\min}}{M_\alpha^{\max} - M_\alpha^{\min}} - \min\left(\frac{Z_\alpha - Z_\alpha^{\min}}{Z_\alpha^{\max} - Z_\alpha^{\min}}, \frac{M_\alpha - M_\alpha^{\min}}{M_\alpha^{\max} - M_\alpha^{\min}}\right) \\
\alpha_4 &:= \min\left(\frac{Z_\alpha - Z_\alpha^{\min}}{Z_\alpha^{\max} - Z_\alpha^{\min}}, \frac{M_\alpha - M_\alpha^{\min}}{M_\alpha^{\max} - M_\alpha^{\min}}\right)
\end{aligned}
\tag{16}
$$

The α_i in Eq. (16) are convex coordinates as they satisfy $0 \le \alpha_i \le 1$ and $\sum_{i=1}^{4} \alpha_i = 1$. Moreover, the parameters Z_α and M_α are known functions of the flight condition of the missile in terms of (α, V, H). Therefore, the dynamic representation of the LPV control structure can be easily updated in real time according to the current dynamics of the missile using Eqs. (15) and (16).

C. Assessment of the LPV Controller

Here, the derived LPV controller is tested and analyzed. Two types of tests can be considered for such a controller structure:

LTI tests. These consist of frequency-domain tests and LTI simulations. In that case the parameter θ is frozen at a single value both for the plant and the controller. The superposition of the step responses of the selected 25 operating conditions is shown in Fig. 7. The specifications in terms of settling time and overshoot are clearly met independently of the parameter value, that is, independently of the LTI stability properties of the plant. The stability margins of the corresponding systems have been evaluated. The gain and phase margins are better than 5 dB and 35 deg, respectively, for the 25 models under consideration. Here, again, the stability margin objectives are satisfied. This good result is mainly due to the coalescence of the Nichols plots in the region of the critical point.

LPV tests. As already stressed, frozen-time tests do not generally ensure that performance and robustness will be preserved when the parameter evolves rapidly. The LPV controller must be further tested for various trajectories $\theta(t)$ in the operating domain. Although LPV tests cannot be exhaustive as they involve an infinite number of possible time-domain functions, three typical parameter trajectories can be investigated.

Trajectory 1. This is a smooth trajectory ranging over the (frozen-time) stable and unstable regions of the missile (see Fig. 8).

Trajectory 2. This trajectory is nonsmooth and is intended to test the reaction of the LPV control law in the face of abrupt parameter changes.

Trajectory 3. This trajectory is of the second type except that the second coordinate of the parameter is corrupted by noise up to 10% of its nominal value. In our opinion, this last test is of fundamental importance for potential real-world implementations of LPV control structures. Indeed, noise or uncertainties are always present on model parameters and any gain-scheduled control law must provide some insensitivity to small parameter fluctuations.

The corresponding LPV simulations 1–3 are presented in Fig. 9. It is seen that the LPV controller performs well and easily withstands switchings between stable, badly damped, and unstable regions of the parameter box without any loss of stability or performance. The settling-time and overshoot specifications are clearly met for trajectory 1. A slight overshoot degradation occurs for trajectory 2. This degradation essentially stems from the corner points of the parameter trajectory where one has an infinite derivative of the parameter. However, from a practical point of view, the response is still satisfying since infinite derivatives are not generally realistic. It has been checked that slightly smoothing the angles of trajectory 2 yields a step response comparable to that of simulation 1. Surprisingly, the overshoot is improved when 10% of relative noise is introduced on the parameter measurement M_α. Accurate measurements of the varying parameter are therefore not required for the LPV controller to be effective. An approximative value indicating the region of the parameter space where the plant is operating is generally sufficient to ensure both robustness and performance of the closed-loop system. Since in practical implementations perfect measurement or estimation is never achieved, this is a very encouraging result.

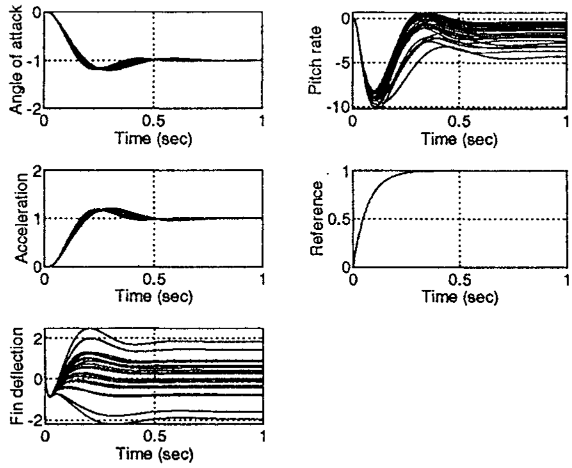

Fig. 7 Simulations of 25 operating points.

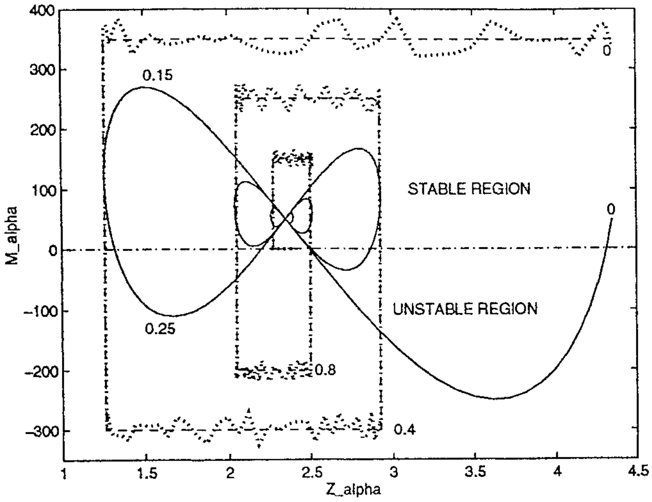

Fig. 8 Parameter trajectories: 1, solid line; 2, dashed line; and 3, dotted line.

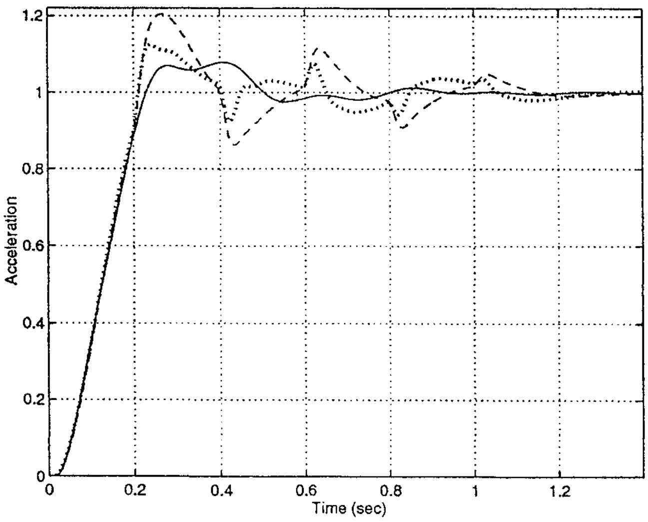

Fig. 9 LPV simulations: 1, solid line; 2, dashed line; and 3, dotted line.

V. Conclusion

In this paper, a synthesis technique for linear parameter-varying plants has been introduced. It is a direct extension of $\mathcal{H}_\infty$ synthesis that additionally incorporates the gain-scheduling task. The following are immediate benefits:

1) There is no need of repeated designs and simulations to handle the global control problem.

2) The time-varying nature of the problem is explicitly taken into account.

A realistic missile autopilot problem has been investigated. This problem is characterized by rapidly varying parameters and the presence of flexible modes conflicting with tight performance specifications. The power and effectiveness of the technique to cope with a large operating range have been demonstrated. Finally, this is a very encouraging result that will contribute to the improvement of traditional approaches to gain scheduling.

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